Homology

Because the physics which goes into determining the structure of main-sequence stars does not change rapidly with mass, one can make the assumption that a star with mass $\mathcal{M}_1$ will just be a scaled version of a star with mass $\mathcal{M}_0$. Stars built under this assumption are called *homologous.* Through homology, relations can be found describing how various stellar quantities (such as radius, luminosity, central temperature, *etc.*) change with mass.

To begin with, consider that if star 1 is really just a scaled up version of star 0, then the radius and mass at any point in star 1 will be

$$r = \left(\frac{R}{R_0}\right) r_0; \qquad \mathcal{M} = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right) \mathcal{M}_0 \tag{17.1}$$

where the subscript 0 refers to the reference star. From the above relations, it is obvious that

$$\frac{dr}{dr_0} = \frac{R}{R_0}; \qquad \frac{d\mathcal{M}}{d\mathcal{M}_0} = \frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} \tag{17.2}$$

With this relation, we can compute the density at any point in the star. From the equation of mass conservation,

$$\frac{dr}{d\mathcal{M}} = \frac{1}{4\pi r^2 \rho} \tag{2.1.3}$$

which implies

$$\frac{dr}{d\mathcal{M}} \left(\frac{dr_0}{dr}\right) \left(\frac{d\mathcal{M}}{d\mathcal{M}_0}\right) = \frac{dr_0}{d\mathcal{M}_0} = \frac{1}{4\pi r^2 \rho} \left(\frac{R_0}{R}\right) \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right) = \frac{1}{4\pi r_0^2 \rho_0}$$

or

$$\frac{\rho}{\rho_0} = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right) \left(\frac{R_0}{R}\right) \left(\frac{r_0^2}{r^2}\right) = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right) \left(\frac{R}{R_0}\right)^{-3} \tag{17.3}$$

Similarly, we can use the equation of momentum conservation and the assumption of hydrostatic equilibrium to compute a relation between pressure, mass, and radius

$$\frac{dP}{d\mathcal{M}}\left(\frac{d\mathcal{M}}{d\mathcal{M}_0}\right)\left(\frac{dP_0}{dP}\right) = -\frac{G\mathcal{M}}{4\pi r^4}\left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)\left(\frac{dP_0}{dP}\right) = -\frac{G\mathcal{M}_0}{4\pi r_0^4}$$

$$\frac{dP}{dP_0} = \left(\frac{\mathcal{M}}{\mathcal{M}_0}\right)\left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)\left(\frac{r_0^4}{r^4}\right) = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)^2\left(\frac{R}{R_0}\right)^{-4}$$

The trivial integration then gives

$$\frac{P}{P_0} = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)^2\left(\frac{R}{R_0}\right)^{-4} \tag{17.4}$$

To make further progress with homology, let's assume that energy generation, opacity, and the equation of state are all power laws of density, pressure, and temperature, and that the power law exponents are the same for every star. Under this assumption, and the definition of the thermodynamic quantities (1.5), (1.6), and (1.7), the stellar densities will be given by

$$\rho \propto P^\alpha T^{-\delta}\mu^\varphi \implies \rho = \rho_0\left(\frac{P}{P_0}\right)^\alpha\left(\frac{T}{T_0}\right)^{-\delta}\left(\frac{\mu}{\mu_0}\right)^\varphi \tag{17.5}$$

Using (11.1) and (11.14), the nuclear reaction rates will be

$$\epsilon \propto \rho^\lambda T^\nu \implies \epsilon = \epsilon_0\left(\frac{\rho}{\rho_0}\right)^\lambda\left(\frac{T}{T_0}\right)^\nu \tag{17.6}$$

and, via (9.13), the opacity can be parameterized by

$$\kappa \propto \rho^p T^q \implies \kappa = \kappa_0\left(\frac{P}{P_0}\right)^s\left(\frac{T}{T_0}\right)^t \tag{17.7}$$

In practice, of course, these exponents will not be the same in every star. For instance, the temperature dependence of nuclear energy generation in a low mass star is significantly different than in a high-mass star. However, for comparisons of similar mass stars, this approximation should be reasonable.

Now consider the equations of mass conservation (2.1.3), pressure balance (2.2.4), energy conservation (2.3.2), and thermal structure under radiative energy transport (2.4.4) and (3.1.6). From (17.3), mass conservation is

$$\left(\frac{\rho}{\rho_0}\right)\left(\frac{R}{R_0}\right)^3 = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)$$

which, through the equation of state (17.5), becomes

$$\left(\frac{P}{P_0}\right)^{\alpha}\left(\frac{T}{T_0}\right)^{-\delta}\left(\frac{\mu}{\mu_0}\right)^{\varphi}\left(\frac{R}{R_0}\right)^3 = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)$$

or

$$3\log\frac{R}{R_0} + \alpha\log\frac{P}{P_0} - \delta\log\frac{T}{T_0} = \log\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} - \varphi\log\frac{\mu}{\mu_0} \tag{17.8}$$

From (17.4), pressure equilibrium is simply

$$4\log\frac{R}{R_0} + \log\frac{P}{P_0} = 2\log\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} \tag{17.9}$$

The relation of energy conservation is also straightforward

$$\frac{d\mathcal{L}}{d\mathcal{M}}\left(\frac{d\mathcal{M}}{d\mathcal{M}_0}\right)\left(\frac{d\mathcal{L}_0}{d\mathcal{L}}\right) = \epsilon\left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)\left(\frac{d\mathcal{L}_0}{d\mathcal{L}}\right) = \epsilon_0$$

If we perform the trivial integration, and substitute (17.6) in for ϵ, then

$$\left(\frac{\epsilon_0}{\epsilon}\right)\left(\frac{\mathcal{L}}{\mathcal{L}_0}\right) = \left(\frac{\rho}{\rho_0}\right)^{-\lambda}\left(\frac{T}{T_0}\right)^{-\nu}\left(\frac{\mathcal{L}}{\mathcal{L}_0}\right) = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)$$

From the equation of state (17.5), we then get

$$\left(\frac{P}{P_0}\right)^{-\lambda\alpha}\left(\frac{T}{T_0}\right)^{\lambda\delta}\left(\frac{\mu}{\mu_0}\right)^{-\lambda\varphi}\left(\frac{T}{T_0}\right)^{-\nu}\left(\frac{\mathcal{L}}{\mathcal{L}_0}\right) = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)$$

or

$$-\lambda\alpha\log\frac{P}{P_0} + (\lambda\delta - \nu)\log\frac{T}{T_0} + \log\frac{\mathcal{L}}{\mathcal{L}_0} = \log\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} + \lambda\varphi\log\frac{\mu}{\mu_0} \tag{17.10}$$

Finally, we can write down the equation for the thermal structure of the (radiative) star

$$\frac{dT}{d\mathcal{M}}\left(\frac{d\mathcal{M}}{d\mathcal{M}_0}\right)\left(\frac{dT_0}{dT}\right) = \frac{3\kappa\mathcal{L}}{64\pi^2 acr^4T^3}\left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)\left(\frac{dT_0}{dT}\right) = \frac{3\kappa_0\mathcal{L}_0}{64\pi^2 acr_0^4T_0^3}$$

When we substitute for r/r_0 using (17.1), and integrate over dT and dT_0, we get

$$\left(\frac{\kappa_0}{\kappa}\right)\left(\frac{\mathcal{L}_0}{\mathcal{L}}\right)\left(\frac{R}{R_0}\right)^4\left(\frac{T}{T_0}\right)^4 = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)$$

If we substitute for the opacity using (17.7), then

$$\left(\frac{P}{P_0}\right)^{-s}\left(\frac{T}{T_0}\right)^{-t}\left(\frac{\mathcal{L}_0}{\mathcal{L}}\right)\left(\frac{R}{R_0}\right)^4\left(\frac{T}{T_0}\right)^4 = \left(\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}\right)$$

or

$$4\log\frac{R}{R_0} - s\log\frac{P}{P_0} + (4-t)\log\frac{T}{T_0} - \log\frac{\mathcal{L}}{\mathcal{L}_0} = \log\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} \tag{17.11}$$

Equations (17.8) - (17.11) now allow us to examine how the radius, pressure, temperature, and luminosity of homologous stars change with mass and chemical composition. First, to study the behavior with mass, set $\mu = \mu_0$. Equations (17.8) - (17.11) in matrix form are then

$$\begin{pmatrix} 3 & \alpha & -\delta & 0 \\ 4 & 1 & 0 & 0 \\ 0 & -\lambda\alpha & (\lambda\delta - \nu) & 1 \\ 4 & -s & (4-t) & -1 \end{pmatrix} \begin{pmatrix} \log\frac{R}{R_0} \\ \log\frac{P}{P_0} \\ \log\frac{T}{T_0} \\ \log\frac{\mathcal{L}}{\mathcal{L}_0} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \log\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} \tag{17.12}$$

Solving these four equations with four unknowns is tedious; the result is

$$\begin{aligned}
\log\frac{R}{R_0} &= \frac{1}{2}(1 + C_1)\ \log\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} \\
\log\frac{P}{P_0} &= -2\,C_1\ \log\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} \\
\log\frac{T}{T_0} &= \frac{1}{2\delta}\left\{1 + (3 - 4\alpha)\,C_1\right\}\ \log\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} \\
\log\frac{\mathcal{L}}{\mathcal{L}_0} &= \left[1 + \frac{4-t}{2\delta} + \left\{2 + 2s + \frac{3-4\alpha}{2\delta}(4-t)\right\} C_1\right]\ \log\frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}
\end{aligned} \tag{17.13}$$

where

$$C_1 = \left\{\frac{4\delta(1 + s + \lambda\alpha)}{\nu + t - 4 - \lambda\delta} + 4\alpha - 3\right\}^{-1}$$

Similarly, by holding the mass constant and allowing μ to vary (*i.e.*, by replacing the right-hand column vector of (17.12) with the vector $[-\varphi, 0, \lambda\varphi, 0]$), one can solve for the dependence of stellar structure on composition:

$$\begin{aligned}
\log \frac{R}{R_0} &= \varphi\, C_2 \log \frac{\mu}{\mu_0} \\
\log \frac{P}{P_0} &= -4\,\varphi\, C_2 \log \frac{\mu}{\mu_0} \\
\log \frac{T}{T_0} &= \frac{\varphi}{\delta} \left\{1 + (3+4\alpha)\, C_2\right\} \log \frac{\mu}{\mu_0} \\
\log \frac{\mathcal{L}}{\mathcal{L}_0} &= \left[\frac{\varphi}{\delta}(4-t) + 2\varphi \left\{2 + 2s + \frac{3-4\alpha}{2\delta}(4-t)\right\} C_2\right] \log \frac{\mu}{\mu_0}
\end{aligned} \tag{17.14}$$

where

$$C_2 = C_1 \left(1 - \frac{\lambda\delta}{\nu + t - 4}\right)^{-1}$$

Note that the solutions to the homology equations are complicated. In general, stellar structure depends on the nuclear reaction rates *and* opacities *and* the equation of state. However, by substituting in for reasonable values of $\alpha, \delta, \lambda, \nu, s$, and t, one can get an idea of how stellar properties should scale with small changes of mass and composition. For example, if $\delta = 0$ (*i.e.*, a polytrope with $n = \alpha/(1-\alpha)$), we have

$$\begin{aligned}
\log \frac{R}{R_0} &= \frac{1}{2}\left(1 + \frac{1}{4\alpha - 3}\right) \log \frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} = \frac{2\alpha - 1}{4\alpha - 3} \log \frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} \\
&= \frac{1-n}{3-n} \log \frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}
\end{aligned}$$

which is the identical expression derived for the mass-radius relation of a polytrope.

HOMOLOGY RELATIONS WITH THE IDEAL GAS LAW

Because homology relations are derived using ratios, the procedure cannot be used if any of the terms are additive. Thus, a general equation of state which include gas and radiation pressure cannot be modeled in this manner. However, if we restrict ourselves to ideal gases, and use reasonable assumptions for the energy generation rate and opacity, then we can use homology to predict stellar structure.

For ideal gases, $\alpha = \delta = \varphi = 1$. Moreover, most nuclear reactions involve two-body collisions, hence the energy generation *per unit volume* goes as ρ^2. Thus, λ, the density coefficient *per unit mass* for nuclear reactions, should also be $\lambda = 1$.

We can now investigate the homology relations for the three opacity laws (electron scattering, Kramers, and H^-), and for values of ν between ~ 5 (for the proton-proton chain) and ~ 20 (for the CNO cycle). The results are shown on the next page.

Dependence of Stellar Structure on Mass

Opacity	$\nu = 5$	$\nu = 10$	$\nu = 15$	$\nu = 20$
$\log R - \log \mathcal{M}$ Coefficient				
e-scattering	0.50	0.69	0.78	0.82
Kramers	0.20	0.52	0.66	0.73
H^-	0.73	0.79	0.82	0.85
$\log P - \log \mathcal{M}$ Coefficient				
e-scattering	0.00	−0.77	−1.11	−1.30
Kramers	1.20	−0.08	−0.63	−0.93
H^-	−0.92	−1.15	−1.30	−1.40
$\log T - \log \mathcal{M}$ Coefficient				
e-scattering	0.50	0.31	0.22	0.17
Kramers	0.80	0.48	0.34	0.27
H^-	0.27	0.21	0.17	0.15
$\log \mathcal{L} - \log \mathcal{M}$ Coefficient				
e-scattering	3.00	3.00	3.00	3.00
Kramers	5.40	5.24	5.17	5.13
H^-	1.16	1.77	2.16	2.43
$\log T_{\mathrm{eff}} - \log \mathcal{M}$ Coefficient				
e-scattering	0.50	0.40	0.36	0.34
Kramers	1.25	1.05	0.96	0.92
H^-	−0.92	0.05	0.13	0.18
$\log \rho - \log \mathcal{M}$ Coefficient				
e-scattering	−0.50	−1.08	−1.33	−1.48
Kramers	0.40	−0.56	−0.98	−1.20
H^-	−1.20	−1.36	−1.47	−1.55

A few items to note:

• For electron scattering, the mass-luminosity relation for stars is independent of the mode of energy generation. The relation is entirely defined by hydrostatic equilibrium, the ideal gas law, and opacity. Thus, stars in this regime will adjust themselves so that the nuclear reactions will provide the requisite amount of luminosity.

• The mass-radius relation for all relevant coefficients is a small positive number. Thus, unlike $n = 3$ polytropic stars (which have a coefficient of $-1/3$), the radius of homologous stars will increase (slightly) with mass.

• Since $T_{\rm eff}$ is formed simply from $\mathcal{L}$ and R, the dependence of $T_{\rm eff}$ on mass is easily found. The coefficient is positive, but the slope of the relation depends on the opacity source. Similarly, we can derive the slope of the relation in the $\log \mathcal{L} - \log T_{\rm eff}$ (HR) diagram.

Opacity	$\nu = 5$	$\nu = 10$	$\nu = 15$	$\nu = 20$
$\log \mathcal{L} - \log T_{\rm eff}$ Coefficient				
e-scattering	6.00	7.43	8.31	8.90
Kramers	4.32	4.99	5.36	5.60

• In a similar manner, we can calculate the behavior of $\mathcal{L}$ and $T_{\rm eff}$ with mean molecular weight, μ. Once again, in the electron scattering case, the dependence of $\mathcal{L}$ on μ is independent of nuclear energy generation. In fact, (17.14) gives $\mathcal{L} \propto \mu^4$; the luminosity is more sensitive to μ than it is to mass!

• We can also calculate the dependence of $T_{\rm eff}$ on μ, and therefore the resulting μ vector in the HR diagram. The resulting slope is shallower than the main sequence, but greater than the lines of constant radii.

Opacity	$\nu = 5$	$\nu = 10$	$\nu = 15$	$\nu = 20$
$\log \mathcal{L} - \log T_{\rm eff}$ Coefficient with μ				
e-scattering	4.27	5.20	5.76	6.13
Kramers	3.43	4.04	4.35	4.53